J. DIFFERENTIAL GEOMETRY
29 (1989) 245-262

MODIFIED DEFECT RELATIONS FOR THE GAUSS MAP OF MINIMAL SURFACES

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Dedicated to Professor Shingo Murakami on his 60th birthday

Introduction

Let $x = (x_1, x_2, x_3) \colon M \to \mathbf{R}^3$ be a connected, oriented immersed minimal surface in $\mathbf{R}^3$. The Gauss map G of M is classically defined to be the map which maps each point p of M to the unit normal vector $G(p) \in S^2$ of M at p. For the sake of convenience, we mean in this paper by the Gauss map of M the map $g \colon M \to \bar{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$ $(= P^1(\mathbf{C}))$ which is the conjugate of the composition of G and the stereographic projection from S^2 onto $\bar{\mathbf{C}}$. By associating a holomorphic local coordinate $z = u + \sqrt{-1}v$ with each positive isothermal coordinate system (u, v), M is considered as a Riemann surface with a conformal metric ds^2. By the assumption of minimality of M, g is a meromorphic function on M.

In 1961, R. Osserman showed that if M is nonflat and complete, then the Gauss map $g \colon M \to \bar{\mathbf{C}}$ cannot omit a set of positive logarithmic capacity [10]. Afterwards, F. Xavier proved that the Gauss map of such a surface can omit at most six points [14]. Recently, the author has shown that the number of exceptional values of the Gauss map of such a surface is at most four [8]. Here, the number four is best-possible. Indeed, there are many kinds of complete minimal surfaces in $\mathbf{R}^3$ whose Gauss maps omit four points ([10] and [12]). The author also obtained some estimate of the Gaussian curvature of a noncomplete minimal surface in $\mathbf{R}^3$ whose Gauss map omits five distinct points [8].

The purpose of this paper is to give some improvements of the above-mentioned results. We shall introduce some new types of modified defects for a nonconstant meromorphic function on an open Riemann surface and give modified defect relations for the Gauss map of a minimal surface in $\mathbf{R}^3$ which have analogy to the defect relation given by R. Nevanlinna in his value distribution theory.

Received August 3, 1987.

1. Statement of the main results

We first give the definitions of modified defects. Let M be an open Riemann surface and f a nonconstant holomorphic map of M into $P^1(\mathbf{C})$. We represent f as $f = (f_0 : f_1)$ with holomorphic functions f_0, f_1 on M without common zero, which we call a reduced representation of f on M in the following. Set $||f|| = (|f_0|^2 + |f_1|^2)^{1/2}$ and, for each $\alpha = (a^0 : a^1) \in P^1(\mathbf{C})$ with $|a^0|^2 + |a^1|^2 = 1$, define the function $F_\alpha := a^1 f_0 - a^0 f_1$.

Definition 1.1. We define the S-defect of α for f by

$$\delta_f^S(\alpha) := 1 - \inf\{\eta \geq 0;\ \eta \text{ satisfies condition } (*)_S\}.$$

Here, condition $(*)_S$ means that there exists a $[-\infty, \infty)$-valued continuous subharmonic function u ($\not\equiv -\infty$) on M satisfying the following conditions:

(D1) $e^u \leq ||f||^\eta$,

(D2) for each $\varsigma \in f^{-1}(\alpha)$ there exists the limit

$$\lim_{z \to \varsigma} (u(z) - \log|z - \varsigma|) \in [-\infty, \infty),$$

where z is a holomorphic local coordinate around ς.

Remark. In the previous papers [6] and [7], we call the S-defect of α the nonintegrated defect of α.

Definition 1.2. We next define the H-defect of α for f by

$$\delta_f^H(\alpha) := 1 - \inf\{\eta \geq 0;\ \eta \text{ satisfies condition } (*)_H\}.$$

Here, condition $(*)_H$ means that there exists a $[-\infty, \infty)$-valued continuous function u on M which is harmonic on $M \backslash f^{-1}(\alpha)$ and satisfies conditions (D1) and (D2).

Definition 1.3. We define also the O-defect of α for f by

$$\delta_f^O(\alpha) := 1 - \inf\{1/m;\ F_\alpha \text{ has no zero of order less than } m\}.$$

Obviously, if η satisfies condition $(*)_H$, then it satisfies condition $(*)_S$. Moreover, if F_α has no zero of order less than m, then $\eta := 1/m$ satisfies condition $(*)_H$. Indeed, the function $u = \eta \log|F_\alpha|$ is harmonic on $M \backslash f^{-1}(\alpha)$ and satisfies conditions (D1) and (D2). From these facts, we see

$$0 \leq \delta_f^O(\alpha) \leq \delta_f^H(\alpha) \leq \delta_f^S(\alpha) \leq 1. \tag{1.4}$$

These modified defects have the following properties similar to those of the classical Nevanlinna defect.

Proposition 1.5. (i) *If there exists a bounded holomorphic function g on M such that $g^{-1}(0) = f^{-1}(\alpha)$, then $\delta_f^H(\alpha) = \delta_f^S(\alpha) = 1$.*

(ii) *If F_α has no zero of order less than m, then*

$$\delta_f^S(\alpha) \geq \delta_f^H(\alpha) \geq \delta_f^O(\alpha) \geq 1 - 1/m.$$

In particular, if $f^{-1}(\alpha) = \emptyset$, then $\delta_f^O(\alpha) = 1$.

Proof. Assertion (ii) is obvious from Definition 1.3. To see (i), we consider the function $u = \log(|g|/K)$, where $K := \sup\{|g(z)|;\, z \in M\}$. Then u satisfies conditions (D1) and (D2) for $\eta = 0$. Thus, $\eta = 0$ satisfies condition $(*)_H$ and so $\delta_f^H(\alpha) = 1$.

We now consider the case where $M = \mathbf{C}$. Without loss of generality, we may assume $f(0) \neq \alpha$. We define the order function of f by

$$T^f(r) := \frac{1}{2\pi}\int_0^{2\pi} \log\|f(re^{i\theta})\|\, d\theta - \log\|f(0)\|,$$

and the counting function for α by

$$N_\alpha^f(r) := \int_0^r \#(f^{-1}(\alpha) \cap \{z\colon |z| \leq t\})\, \frac{dt}{t},$$

where $\#A$ denotes the number of elements of a set A. Then the classical Nevanlinna defect without counted multiplicities is defined by

$$\delta_f(\alpha) := 1 - \limsup_{r\to\infty} \frac{N_\alpha^f(r)}{T^f(r)}.$$

By the help of Jensen's formula, we can show easily

$$0 \leq \delta_f^S(\alpha) \leq \delta_f(\alpha), \tag{1.6}$$

[6, Proposition 4.7].

Now, we state our main results. First, we give

Theorem I. *Let $x\colon M \to \mathbf{R}^3$ be a nonflat complete minimal surface and $g\colon M \to P^1(\mathbf{C})$ the Gauss map. Then, for arbitrarily given distinct points $\alpha_1, \cdots, \alpha_q \in P^1(\mathbf{C})$,*

$$\sum_{j=1}^{q} \delta_g^H(\alpha_j) \leq 4.$$

Since we have $\delta_g^H(\alpha_j) = 1$ for every $\alpha_j \notin g(M)$ by Proposition 1.5, Theorem I yields the following result which was given in [8].

Corollary 1.7. *The Gauss map of a nonflat complete minimal surface in $\mathbf{R}^3$ can omit at most four points.*

We next consider a noncomplete minimal surface $x\colon M \to \mathbf{R}^3$. We denote by $d(p)$ the distance from a point $p \in M$ to the boundary of M, namely, the largest lower bound of the lengths of all piecewise smooth curves going from p to the boundary of M, and by $K(p)$ the Gaussian curvature of M at p.

Theorem II. *Let $x\colon M \to \mathbf{R}^3$ be a nonflat noncomplete minimal surface and g the Gauss map. If there exist distinct points $\alpha_1, \cdots, \alpha_q \in P^1(\mathbf{C})$ such that $\sum_{j=1}^q \delta_g^O(\alpha_j) > 4$, then*

$$|K(p)| \leq C/d(p)^2$$

for all $p \in M$, *where* C *is a positive constant depending only on* $\alpha_1, \cdots, \alpha_q$ *and* $\delta_g^O(\alpha_1), \cdots, \delta_g^O(\alpha_q)$.

This is an improvement of [8, Theorem I].

Let $x\colon M \to \mathbf{R}^4$ be a minimal surface in $\mathbf{R}^4$. As is well known, the set of all oriented 2-planes in $\mathbf{R}^4$ is canonically identified with the quadric

$$Q_2(\mathbf{C}) = \{(w_1 : \cdots : w_4) \in P^3(\mathbf{C});\ w_1^2 + w_2^2 + w_3^2 + w_4^2 = 0\}$$

in $P^3(\mathbf{C})$. The Gauss map of M is defined by the map $G\colon M \to Q_2(\mathbf{C})$ which maps each point $p \in M$ to the point $G(p) \in Q_2(\mathbf{C})$ corresponding to the oriented tangent plane of M at p. Since $Q_2(\mathbf{C})$ is canonically biholomorphic with $P^1(\mathbf{C})\times P^1(\mathbf{C})$, G may be identified with a pair of meromorphic functions $g = (g_1, g_2)\colon M \to P^1(\mathbf{C}) \times P^1(\mathbf{C})$. We can prove the following.

Theorem III. *Let* $x\colon M \to \mathbf{R}^4$ *be a complete minimal surface and* $g = (g_1, g_2)\colon M \to P^1(\mathbf{C}) \times P^1(\mathbf{C})$ *the Gauss map of* M.

(i) *Assume that* $g_1 \not\equiv$ const. *and* $g_2 \not\equiv$ const. *Then, for arbitrary distinct* $\alpha_{11}, \cdots, \alpha_{1q_1} \in P^1(\mathbf{C})$ *and distinct* $\alpha_{21}, \cdots, \alpha_{2q_2} \in P^1(\mathbf{C})$, *at least one of the following conclusions is valid*:

$$\text{(a)}\quad \sum_{i=1}^{q_1} \delta_{g_1}^H(\alpha_{1i}) \le 2,$$

$$\text{(b)}\quad \sum_{j=1}^{q_2} \delta_{g_2}^H(\alpha_{2j}) \le 2,$$

$$\text{(c)}\quad \frac{1}{\sum_{i=1}^{q_1} \delta_{g_1}^H(\alpha_{1i}) - 2} + \frac{1}{\sum_{j=1}^{q_2} \delta_{g_2}^H(\alpha_{2j}) - 2} \ge 1.$$

(ii) *Assume that* $g_1 \not\equiv$ const. *and* $g_2 \equiv$ const. *Then, for arbitrary distinct points* $\alpha_1, \cdots, \alpha_q \in P^1(\mathbf{C})$, *we have*

$$\sum_{j=1}^{q} \delta_{g_1}^H(\alpha_j) \le 3.$$

This is an improvement of Theorem II of [8].

After giving the Main Lemma in the next section, we shall prove Theorems I, II and III in §§3, 4 and 5 respectively.

2. Main Lemma

Let f be a nonconstant holomorphic map of a disc $\Delta_R := \{z \in \mathbf{C};\ |z| < R\}$ into $P^1(\mathbf{C})$, where $0 < R < \infty$. Take a reduced representation $f = (f_0 : f_1)$ on Δ_R and define

$$||f|| := (|f_0|^2 + |f_1|^2)^{1/2}, \qquad W(f_0, f_1) := f_0 f_1' - f_1 f_0'.$$

For arbitrarily given q distinct points $\alpha_j = (a_j^0 : a_j^1)$ $(1 \leq j \leq q)$, set

$$F_j := a_j^1 f_0 - a_j^0 f_1 \qquad (1 \leq j \leq q),$$

where $|a_j^0|^2 + |a_j^1|^2 = 1$.

Proposition 2.1. *For each $\varepsilon > 0$ there exist positive constants C and μ depending only on $\alpha_1, \cdots, \alpha_q$ and on ε respectively such that*

$$\Delta \log \left(\frac{||f||^{\varepsilon}}{\prod_{j=1}^{q} \log(\mu ||f||^2/|F_j|^2)} \right) \geq C \frac{||f||^{2q-4} |W(f_0, f_1)|^2}{\prod_{j=1}^{q} |F_j|^2 \log^2(\mu ||f||^2/|F_j|^2)}.$$

This is a restatement of a special case of [4, §6, Proposition] (cf. [13, §6]). For the sake of completeness of self-containedness, we give here a direct proof. We show first

Lemma 2.2. *For each $\varepsilon > 0$ there exists a constant $\mu_0(\varepsilon) \geq 1$ such that, for every $\mu \geq \mu_0(\varepsilon)$,*

$$\Delta \log \frac{1}{\log(\mu ||f||^2/|F_j|^2)} \geq \frac{4|W(f_0, f_1)|^2}{||f||^2 |F_j|^2 \log^2(\mu ||f||^2/|F_j|^2)} - \varepsilon \Delta \log ||f||^2.$$

Proof. Set $\varphi_j := |F_j|^2/||f||^2$. We have

$$\begin{aligned}
\left| \frac{\partial \varphi_j}{\partial z} \right|^2 &= \frac{1}{||f||^8} \left| F_j' \bar{F}_j ||f||^2 - |F_j|^2 (f_0' \bar{f}_0 + f_1' \bar{f}_1) \right|^2 \\
&= \frac{|F_j|^2}{||f||^8} |W(f_0, f_1)|^2 |a_j^0 \bar{f}_0 + a_j^1 \bar{f}_1|^2 \\
&= \frac{|F_j|^2}{||f||^8} |W(f_0, f_1)|^2 \left((|a_j^0|^2 + |a_j^1|^2)(|f_0|^2 + |f_1|^2) - |a_j^1 f_0 - a_j^0 f_1|^2 \right) \\
&= (\varphi_j - \varphi_j^2) \frac{|W(f_0, f_1)|^2}{||f||^4}.
\end{aligned}$$

On the other hand, it holds that

$$\begin{aligned}
\frac{\partial^2 \log ||f||^2}{\partial z \partial \bar{z}} &= \frac{(|f_0'|^2 + |f_1'|^2)(|f_0|^2 + |f_1|^2) - |f_0 \bar{f}_0' + f_1 \bar{f}_1'|^2}{||f||^4} \\
&= \frac{|W(f_0, f_1)|^2}{||f||^4}.
\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta \log \frac{1}{\log(\mu/\varphi_j)} &= \frac{4}{\log(\mu/\varphi_j)} \frac{\partial^2 \log \varphi_j}{\partial z \partial \bar{z}} + \frac{4}{\varphi_j^2 \log^2(\mu/\varphi_j)} \left| \frac{\partial \varphi_j}{\partial z} \right|^2 \\ &= -\frac{4}{\log(\mu/\varphi_j)} \frac{\partial^2 \log ||f||^2}{\partial z \partial \bar{z}} \\ &\quad + \frac{4(\varphi_j - \varphi_j^2)}{\varphi_j^2 \log^2(\mu/\varphi_j)} \frac{\partial^2 \log ||f||^2}{\partial z \partial \bar{z}} \\ &= \frac{4}{\varphi_j \log^2(\mu/\varphi_j)} \frac{|W(f_0, f_1)|^2}{||f||^4} \\ &\quad - 4\left(\frac{1}{\log^2(\mu/\varphi_j)} + \frac{1}{\log(\mu/\varphi_j)} \right) \frac{\partial^2 \log ||f||^2}{\partial z \partial \bar{z}}.\end{aligned}$$

If we choose a positive constant $\mu_0(\varepsilon)$ with

$$\frac{1}{\log^2 \mu_0(\varepsilon)} + \frac{1}{\log \mu_0(\varepsilon)} < \varepsilon,$$

we have the desired inequality because $|\varphi_j| \leq 1$.

Proof of Proposition 2.1. For a given $\varepsilon > 0$ we take a constant μ with $\mu \geq \mu_0(\varepsilon/q)$. By Lemma 2.2, we obtain

$$\begin{aligned}&\Delta \log \frac{||f||^\varepsilon}{\prod_{j=1}^q \log(\mu ||f||^2/|F_j|^2)} \\ &\geq \varepsilon \cdot \Delta \log ||f||^2 + \sum_{j=1}^q \left(\frac{4|W(f_0, f_1)|^2}{||f||^2 |F_j|^2 \log^2(\mu ||f||^2/|F_j|^2)} - \frac{\varepsilon}{q} \Delta \log ||f||^2 \right) \\ &= \frac{4|W(f_0, f_1)|^2}{||f||^4} \sum_{j=1}^q \frac{||f||^2}{|F_j|^2 \log^2(\mu ||f||^2/|F_j|^2)}.\end{aligned}$$

On the other hand, for each (i, j) with $1 \leq i < j \leq q$, there exists a constant C_{ij} depending only on α_i and α_j such that

$$||f|| \leq C_{ij} \max(|F_i|, |F_j|),$$

because f_0 and f_1 can be represented as a linear combination of F_i and F_j. Set $C_0 := \max_{1 \leq i < j \leq q} C_{ij}$ and

$$M := \max\{x/\log^2 \mu x;\ 1 < x \leq C_0^2\}.$$

For an arbitrarily fixed $z \in \Delta_R$ we determine indices $j_1, \cdots, j_q$ with $\{j_1, \cdots, j_q\} = \{1, 2, \cdots, q\}$ so that

$$|F_{j_1}(z)| \leq |F_{j_2}(z)| \leq \cdots \leq |F_{j_q}(z)|.$$

Then, for $l = 2, 3, \cdots, q$, we have $||f(z)|| \leq C_0 |F_{j_l}(z)|$ and so

$$\frac{||f(z)||^2}{|F_{j_l}(z)|^2 \log^2(\mu ||f(z)||^2 / |F_{j_l}|^2)} \leq M.$$

Therefore, at the point z, we obtain

$$\begin{aligned}
&\sum_{j=1}^{q} \frac{||f||^2}{|F_j|^2 \log^2(\mu ||f||^2 / |F_j|^2)} \\
&\geq \frac{||f||^2}{|F_{j_1}|^2 \log^2(\mu ||f||^2 / |F_{j_1}|^2)} \\
&\geq \frac{1}{M^{q-1}} \left(\prod_{l=2}^{q} \frac{||f||^2}{|F_{j_l}|^2 \log^2(\mu ||f||^2 / |F_{j_l}|^2)} \right) \frac{||f||^2}{|F_{j_1}|^2 \log^2(\mu ||f||^2 / |F_{j_1}|^2)} \\
&= \frac{||f||^{2q}}{M^{q-1} \prod_{j=1}^{q} |F_j|^2 \log^2(\mu ||f||^2 / |F_j|^2)}.
\end{aligned}$$

Since the last term does not depend on choices of indices $j_1, \cdots, j_q$, this holds on the totality of Δ_R. Combining this with the inequality obtained above, we conclude Proposition 2.1.

Now, we consider $[-\infty, \infty)$-valued continuous subharmonic functions u_j ($\not\equiv -\infty$) on Δ_R and nonnegative numbers η_j $(1 \leq j \leq q)$ satisfying the conditions:

(C1) $\gamma := q - 2 - (\eta_1 + \cdots + \eta_q) > 0$,
(C2) $e^{u_j} \leq ||f||^{\eta_j}$ for $j = 1, 2, \cdots, q$,
(C3) for each $\varsigma \in f^{-1}(\alpha_j)$ $(1 \leq j \leq q)$ there exists the limit

$$\lim_{z \to \varsigma} (u_j(z) - \log|z - \varsigma|) \in [-\infty, \infty).$$

Lemma 2.3. *For positive constants C and μ (> 1), set*

$$v := C \frac{||f||^{\gamma} e^{u_1 + \cdots + u_q} |W(f_0, f_1)|}{\prod_{j=1}^{q} |F_j| \log(\mu ||f||^2 / |F_j|^2)}$$

on $\Delta_R \backslash \{F_1 \ldots F_q = 0\}$ and $v := 0$ on $\Delta_R \cap \{F_1 \ldots F_q = 0\}$. Then v is continuous on Δ_R and satisfies the condition $\Delta \log v \geq v^2$ in the distribution sense for suitably chosen C, μ depending only on α_j and η_j $(1 \leq j \leq q)$.

Proof. Obviously, v is continuous on $\{F_1 \ldots F_q \neq 0\}$. Take a point ς with $F_i(\varsigma) = 0$ for some i. Then $F_j(\varsigma) \neq 0$ for all $j \neq i$. Changing indices if necessary, we may assume that $f_0(\varsigma) \neq 0$. Set $\chi_i := W(f_0, f_1)/F_i$. It has a pole of order one at ς because we can write $\chi_i = -(f_0/a_i^0)(g'/(g - \alpha_i))$ for $g := f_1/f_0$. Therefore, the function

$$\frac{e^{u_i} |W(f_0, f_1)|}{|F_i|} = (|z - \varsigma|\, |\chi_i|) e^{u_i - \log|z - \varsigma|}$$

is bounded in a neighborhood of ς. This implies that $\lim_{z\to\varsigma} v(z) = 0$. Eventually, v is continuous on Δ_R.

Now, we choose constants C and μ such that C^2 and μ satisfy the inequality in Proposition 2.1 for the case $\varepsilon = \gamma$. We then have

$$\begin{aligned}\Delta \log v &\geq \Delta \log \frac{||f||^{\gamma}}{\prod_{j=1}^{q} \log(\mu||f||^2/|F_j|^2)} \\ &\geq C^2 \frac{||f||^{2q-4}|W(f_0, f_1)|^2}{\prod_{j=1}^{q} |F_j|^2 \log^2(\mu||f||^2/|F_j|^2)} \\ &\geq C^2 \frac{||f||^{2\gamma} e^{2(u_1+\cdots+u_q)}|W(f_0, f_1)|^2}{\prod_{j=1}^{q} |F_j|^2 \log^2(|\mu||f||^2/|F_j|^2)} \\ &= v^2.\end{aligned}$$

Lemma 2.4. *For the above u_j, η_j and γ, we can choose positive constants C^* and μ such that*

$$\frac{||f||^{\gamma} e^{u_1+\cdots+u_q}|W(f_0, f_1)|}{\prod_{j=1}^{q} |F_j| \log(\mu||f||^2/|F_j|^2)} \leq C^* \frac{2R}{R^2 - |z|^2}.$$

This is an immediate consequence of Lemma 2.3 and the following generalized Schwarz' Lemma.

Lemma 2.5 (*cf.* [1]). *Let v be a nonnegative real-valued continuous subharmonic function on Δ_R. If v satisfies the inequality $\Delta \log v \geq v^2$ in the distribution sense, then*

$$v(z) \leq \lambda_R(z) := \frac{2R}{R^2 - |z|^2}.$$

Proof. Since $\lambda_r(z)$ is continuous in r, we have only to show that

$$\eta_r(z) := v(z)/\lambda_r(z) \leq 1$$

on Δ_r for every $r < R$. Since $\lim_{z\to\partial\Delta_r} \eta_r(z) = 0$, there exists a point $z_0 \in \Delta_r$ such that $\eta_r(z_0) = \max\{\eta_r(z);\ z \in \bar{\Delta}_r\}$. Suppose that $\eta_r(z_0) > 1$. Then $\eta_r(z) > 1$ and so $v(z) > \lambda_r(z)$ on an open neighborhood U of z_0. By the assumption,

$$\Delta \log \eta_r = \Delta \log v - \Delta \log \lambda_r \geq v^2 - \lambda_r^2 > 0 \tag{2.6}$$

in the distribution sense on U. Therefore $\log \eta_r$ is subharmonic and necessarily a constant on U by the maximum principle. This contradicts (2.6). Thus $\eta_r(z_0) \leq 1$ and so $\eta_r(z) \leq 1$ on Δ_r.

We now give the

Main Lemma. *Let $u_1, \cdots, u_q$ be continuous subharmonic functions on M, and $\eta_1, \cdots, \eta_q$ nonnegative constants which satisfy the conditions* (C1)–(C3). *Then, for every δ with $0 < q\delta < \gamma$, there exists a constant C_0 such that*

$$\frac{||f||^{\gamma-q\delta}e^{u_1+\cdots+u_q}|W(f_0,f_1)|}{|F_1F_2\ldots F_q|^{1-\delta}} \leq C_0\frac{2R}{R^2-|z|^2}. \tag{2.7}$$

Proof. For a given δ we set

$$\tilde{C} := \sup_{0<x\leq 1} x^\delta \log(\mu/x^2)(< +\infty).$$

Then we have

$$\begin{aligned}&\frac{||f||^{\gamma-q\delta}e^{u_1+\cdots+u_q}|W(f_0,f_1)|}{|F_1F_2\ldots F_q|^{1-\delta}}\\ &= \frac{||f||^{\gamma}e^{u_1+\cdots+u_q}|W(f_0,f_1)|}{|F_1F_2\ldots F_q|}\prod_{j=1}^{q}\left(\frac{|F_j|}{||f||}\right)^\delta\\ &\leq \tilde{C}^q\frac{||f||^{\gamma}e^{u_1+\cdots+u_q}|W(f_0,f_1)|}{\prod_{j=1}^{q}|F_j|\log(\mu||f||^2/|F_j|^2)}\\ &\leq C^*\tilde{C}^q\left(\frac{2R}{R^2-|z|^2}\right),\end{aligned}$$

where C^* and μ are the constants given in Lemma 2.4. This gives the Main Lemma.

We later need the following modified defect relation which is a direct result of the classical Nevanlinna defect relation and (1.6). We give here a direct proof of this by the use of the Main Lemma.

Theorem 2.8. *Let $f\colon \mathbf{C} \to P^1(\mathbf{C})$ be a nonconstant holomorphic map. For arbitrary distinct points $\alpha_1, \cdots, \alpha_q \in P^1(\mathbf{C})$*

$$\sum_{j=1}^{q}\delta_f^S(\alpha_j) \leq 2.$$

Proof. Without loss of generality, we may assume $u_j(0) \neq -\infty$, $f(0) \neq \alpha_j$ $(1 \leq j \leq q)$ and $W(f_0, f_1)(0) \neq 0$, where f_0, f_1 are holomorphic functions on $\mathbf{C}$ such that $f = (f_0 : f_1)$ is a reduced representation. Suppose that $\sum_{j=1}^q \delta_f^S(\alpha_j) > 2$. Then there exist positive constants $\eta_1, \cdots, \eta_q$ satisfying condition (C1) and continuous subharmonic functions $u_1, \cdots, u_q$ on M satisfying conditions (C2) and (C3). For every $R > 0$ and δ with $\gamma > q\delta > 0$ we apply the Main Lemma to the map $f|\Delta_R\colon \Delta_R \to P^1(\mathbf{C})$. Substitute $z = 0$

into inequality (2.7). We can conclude that R is bounded by a constant depending only on α_j, η_j and the values of f, u_j, F_j, $W(f_0, f_1)$ at the origin. This is a contradiction. Thus, we have Theorem 2.8.

3. Proof of Theorem I

Let $x = (x_1, x_2, x_3)\colon M \to \mathbf{R}^3$ be a nonflat minimal surface and $g\colon M \to P^1(\mathbf{C})$ the Gauss map. The argument in this section is also used for the proof of Theorems II and III. We do not assume completeness of M for the present. For our purpose, we may assume that M is simply connected. In fact, for the universal covering surface $\pi\colon \tilde{M} \to M$, $\tilde{x} := x \cdot \pi\colon \tilde{M} \to \mathbf{R}^3$ is also a nonflat minimal surface, and complete if M is complete. Moreover, the Gauss map of $\tilde{M}$ is given by $\tilde{g} := g \cdot \pi$, and the modified defects for g are not larger than those for $\tilde{g}$. Since there is no compact minimal surface in $\mathbf{R}^3$, M is biholomorphic with $\mathbf{C}$ or the unit disc in $\mathbf{C}$. For the case $M = \mathbf{C}$, Theorem I is true by virtue of Theorem 2.8. In the following, we assume that M is biholomorphic with the unit disc in $\mathbf{C}$.

Set $\phi_i := \partial x_i/\partial z$ $(i = 1, 2, 3)$ and $f := \phi_1 - \sqrt{-1}\phi_2$. Then, the Gauss map $g\colon M \to P^1(\mathbf{C})$ is given by

$$g = \phi_3/(\phi_1 - \sqrt{-1}\phi_2),$$

and the metric on M induced from $\mathbf{R}^3$ is given by

$$ds^2 = |f|^2(1 + |g|^2)^2\,|dz|^2, \tag{3.1}$$

[12]. Take a reduced representation $g = (g_0 : g_1)$ on M and set $\|g\| = (|g_0|^2 + |g_1|^2)^{1/2}$. Then we can rewrite

$$ds^2 = |h|^2\|g\|^4\,|dz|^2,$$

where $h := f/g_0^2$.

Now, for given q distinct points $\alpha_1, \cdots, \alpha_q \in P^1(\mathbf{C})$ we assume that

$$\sum_{j=1}^{q} \delta_g^H(\alpha_j) > 4. \tag{3.2}$$

By Definition 1.2, there exist constants $\eta_j \geq 0$ $(1 \leq j \leq q)$ such that $\gamma := q - 2 - (\eta_1 + \cdots + \eta_q) > 2$ and continuous functions u_j $(1 \leq j \leq q)$ on M such that each u_j is harmonic on $M \backslash f^{-1}(\alpha_j)$ and satisfies conditions (C2) and (C3). Take δ with

$$(\gamma - 2)/q > \delta > (\gamma - 2)/(q + 2), \tag{3.3}$$

and set $p = 2/(\gamma - q\delta)$. Then

$$0 < p < 1, \qquad \delta p/(1 - p) > 1. \tag{3.4}$$

Set $M' := M \backslash \{F_1 F_2 \dots F_q W(g_0, g_1) = 0\}$ and define the function

$$v := |h|^{1/(1-p)} \left(\frac{|F_1 F_2 \dots F_q|^{1-\delta}}{e^{u_1 + \cdots + u_q} |W(g_0, g_1)|} \right)^{p/(1-p)} \tag{3.5}$$

on M', where $F_j := a_j^1 g_0 - a_j^0 g_1$ for representations $\alpha_j = (a_j^0 : a_j^1)$ with $|a_j^0|^2 + |a_j^1|^2 = 1$ $(1 \leq j \leq q)$. Let $\pi \colon \tilde{M}' \to M'$ be the universal covering surface of M'. By the assumption, $\log v \cdot \pi$ is harmonic on $\tilde{M}'$. Take a conjugate harmonic function v^* of $\log v \cdot \pi$ on $\tilde{M}'$ and define the holomorphic function $\psi := e^{\log v \cdot \pi + i v^*}$, which satisfies the identity $|\psi| = v \cdot \pi$. Choose a point $o \in M'$. We may regard o as the origin in $\mathbf{C}$. Each $\tilde{z}$ of $\tilde{M}'$ corresponds bijectively to the homotopy class of a continuous curve $\gamma_{\tilde{z}} \colon [0,1] \to M'$ and $\gamma_{\tilde{z}}(0) = o$ and $\gamma_{\tilde{z}}(1) = \pi(\tilde{z})$. We denote by $\tilde{o}$ the point corresponding to the constant curve o. Set

$$w = F(\tilde{z}) = \int_{\gamma_{\tilde{z}}} \psi(z)\, dz.$$

Then, F is a single-valued holomorphic function on $\tilde{M}'$ and satisfies the conditions $F(\tilde{o}) = 0$ and $dF(\tilde{z}) \neq 0$ for every $\tilde{z} \in \tilde{M}'$. Therefore, F maps an open neighborhood U of $\tilde{o}$ biholomorphically onto an open disc $\Delta_R := \{w \colon |w| < R\}$ in $\mathbf{C}$, where $0 < R \leq +\infty$. Choose the largest R with this property and define $\Phi := \pi \cdot (F|U)^{-1}$. Then $R < +\infty$ because of Liouville's theorem.

We now consider the line segment

$$L_a \colon w = ta, \qquad 0 \leq t < 1,$$

in Δ_R and the image

$$\Gamma_a \colon z = \Phi(ta), \qquad 0 \leq t < 1,$$

of L_a by Φ for each point $a \in \partial \Delta_R$. We claim that there exists a point $a_0 \in \partial \Delta_R$ such that Γ_{a_0} tends to the boundary of M. Assume the contrary. Then, for each $a \in \partial \Delta_R$ there is a sequence $\{t_\nu;\ \nu = 1, 2, \dots\}$ such that $\lim_{\nu \to \infty} t_\nu = 1$ and $z_0 := \lim_{\nu \to \infty} \Phi(t_\nu a)$ exists in M. Suppose that $z_0 \notin M'$. Then z_0 is a zero of one of the holomorphic functions $F_1, \cdots, F_q$ and $W(g_0, g_1)$. By the same argument as in the proof of Lemma 2.3, it can be shown that

$$\liminf_{z \to z_0} |(F_1 F_2 \dots F_q)(z)|^{\delta p/(1-p)} v(z) > 0$$

in the case $F_i(z_0) = 0$ for some i, and

$$\liminf_{z \to z_0} |W(g_0, g_1)(z)|^{p/(1-p)} v(z) > 0$$

in the case $W(g_0, g_1)(z_0) = 0$. In any case, we can find a positive constant C such that $v \geq C/|z - z_0|^{\delta p/(1-p)}$ in a neighborhood of z_0. By virtue of (3.4), we get

$$R = \int_{L_a} |dw| = \int_{\Gamma_a} \left| \frac{dw}{dz} \right| |dz| = \int_{\Gamma_a} v(z) \, |dz|$$
$$\geq C \int_{\Gamma_a} \frac{1}{|z - z_0|^{\delta p/(1-p)}} |dz| = \infty.$$

This is a contradiction. Therefore, $z_0 \in M'$.

Take a simply connected neighborhood V of z_0, which is relatively compact in M'. Since v is positive continuous, we have $C' := \min_{z \in \bar{V}} v(z) > 0$. If there exists a sequence $\{t'_\nu; \nu = 1, 2, \dots\}$ such that $\lim_{\nu \to \infty} t'_\nu = 1$ and $\Phi(t'_\nu a) \notin V$, then Γ_a goes and returns infinitely often from ∂V to a sufficiently small neighborhood of z_0, and so we have an absurd conclusion

$$R = \int_{L_a} |dw| \geq C' \int_{\Gamma_a} |dz| = \infty.$$

Therefore, $\Phi(ta) \in V$ $(t_0 < t < 1)$ for some t_0. Moreover, since V can be replaced by an arbitrarily small neighborhood of z_0 in the above argument, we can conclude that $\lim_{t \to 1} \Phi(ta) = z_0$. Let $\tilde{V}$ be a connected component of $\pi^{-1}(V)$, which includes $\{(F|U)^{-1}(ta);\ t_0 < t < 1\}$. Since $\pi|\tilde{V} : \tilde{V} \to V$ is a homeomorphism, there exists the limit

$$\tilde{z}_0 := \lim_{t \to 1} (F|U)^{-1}(ta) \in \tilde{M}'.$$

Then F maps an open neighborhood of $\tilde{z}_0$ biholomorphically onto a neighborhood of a. Eventually, $(F|U)^{-1}$ has a holomorphic extension to a neighborhood of each $a \in \partial \Delta_R$ as a map into $\tilde{M}'$. Since $\partial \Delta_R$ is compact, we can easily find a constant R' with $R < R'$ such that F maps an open neighborhood of $\bar{U}$ biholomorphically onto $\Delta_{R'}$. This contradicts the property of R. Therefore, there exists a point $a_0 \in \partial \Delta_R$ such that Γ_{a_0} tends to the boundary of M.

The map $z = \Phi(w)$ is locally biholomorphic, and the metric on M' induced from ds^2 through Φ is given by

$$\Phi^* \, ds^2 = |h \circ \Phi|^2 ||g \circ \Phi||^4 \left| \frac{dz}{dw} \right|^2 |dw|^2.$$

On the other hand, by the definition of $w = F(z)$ we have, because of (3.1),

$$\left| \frac{dw}{dz} \right|^{1-p} = \frac{|h| \, |F_1 F_2 \dots F_q|^{(1-\delta)p}}{(e^{u_1 + \dots + u_q} |W(g_0, g_1)|)^p}.$$

Set $f := g \circ \Phi$, $f_0 = g_0 \circ \Phi$, $f_1 = g_1 \circ \Phi$ and abbreviate $u_j \circ \Phi$ and $F_j \circ \Phi$ by u_j and F_j respectively. Since

$$W(f_0, f_1) = (W(g_0, g_1) \circ \Phi) \frac{dz}{dw},$$

we obtain

$$\left|\frac{dz}{dw}\right| = \frac{(e^{u_1+\cdots+u_q}|W(f_0, f_1)|)^p}{|h|\,|F_1F_2\ldots F_q|^{(1-\delta)p}}.$$

Therefore,

$$\Phi^* \, ds^2 = \left(\frac{||f||^2(e^{u_1+\cdots+u_q}|W(f_0, f_1)|)^p}{|F_1F_2\ldots F_q|^{(1-\delta)p}}\right)^2 |dw|^2. \tag{3.6}$$

We apply here the Main Lemma to the map $f\colon \Delta_R \to P^1(\mathbf{C})$ to see

$$\Phi^* \, ds^2 \leq C_0^{2p}\left(\frac{2R}{R^2-|w|^2}\right)^{2p} |dw|^2.$$

It then follows that

$$\begin{aligned} d(0) &\leq \int_{\Gamma_{a_0}} ds = \int_{L_{a_0}} \Phi^* \, ds \\ &\leq C_0^p \int_0^R \left(\frac{2R}{R^2-|w|^2}\right)^p |dw| = C_1 R^{1-p}, \end{aligned} \tag{3.7}$$

where C_0 and C_1 are positive constants depending only on α_j and $\delta_g^H(\alpha_j)$ $(\leq \delta_f^H(\alpha_j))$.

Now, as in Theorem I, suppose that M is complete. Then $d(0) = \infty$. This contradicts the fact $R < \infty$. For a nonflat complete minimal surface in $\mathbf{R}^3$, (3.2) is not true. This completes the proof of Theorem I.

4. Proof of Theorem II

As in Theorem II, let $x\colon M \to \mathbf{R}^3$ be a nonflat minimal surface, and $g\colon M \to P^1(\mathbf{C})$ be the Gauss map, and assume that

$$\sum_{j=1}^{q} \delta_g^O(\alpha_j) > 4, \tag{4.1}$$

for q distinct points $\alpha_1, \cdots, \alpha_q \in P^1(\mathbf{C})$. For our purpose, we may assume that M is biholomorphic with the unit disc in $\mathbf{C}$. We use the same notation as in the previous section. By Definition 1.3, there exist positive integers $m_1, \cdots, m_q$ such that

$$\gamma := \left(1-\frac{1}{m_1}\right) + \cdots + \left(1-\frac{1}{m_q}\right) - 2 > 2,$$

and each F_j $(1 \leq j \leq q)$ has no zero of order less than m_j. Set $\eta_j := 1/m_j$ and $u_j := \eta_j \log|F_j|$. Thus, u_j are harmonic on $M \backslash f^{-1}(\alpha_j)$ and satisfy conditions (C2) and (C3) in §2 for the map $g\colon M \to P^1(\mathbf{C})$. All arguments in the previous section work for the constants η_j and functions u_j $(1 \leq j \leq q)$. By the same method as in the previous section, we can define a holomorphic map

$$\Phi\colon \Delta_R \to M' := M \backslash \{F_1 F_2 \dots F_q W(g_0, g_1) = 0\},$$

such that the induced metric on Δ_R is given by (3.6) and satisfies condition (3.7), where $f = (f_0 : f_1) = g \circ \Phi$.

Now, apply the Main Lemma to the map f to show that

$$\frac{||f||^{\gamma - q\delta} |W(f_0, f_1)|}{|F_1|^{1-\eta_1-\delta} \dots |F_q|^{1-\eta_q-\delta}} = \frac{||f||^{\gamma - q\delta} e^{u_1 + \dots + u_q} |W(f_0, f_1)|}{|F_1 F_2 \dots F_q|^{1-\delta}} \leq C_0 \left(\frac{2R}{R^2 - |w|^2} \right),$$

where $0 < q\delta < \gamma$, and C_0 is a constant depending only on α_j and η_j. Set $p = 2/(\gamma - q\delta)$ and substitute $w = 0$ into this inequality. We can conclude

$$R^{1-p} \leq (2C_0)^{1-p} \frac{(|F_1(0)|^{1-\eta_1-\delta} \dots |F_q(0)|^{1-\eta_q-\delta})^{1-p}}{|W(f_0, f_1)(0)|^{1-p} ||f(0)||^{2(1-p)/p}}. \tag{4.2}$$

On the other hand, by substituting $e^{u_j} = |F_j|^{\eta_j}$ into the identity (3.6), we obtain

$$\Phi^* ds^2 = \lambda^2 |dw|^2 = \frac{||f||^4 |W(f_0, f_1)|^{2p}}{(|F_1|^{1-\eta_1-\delta} \dots |F_q|^{1-\eta_q-\delta})^{2p}} |dw|^2.$$

Therefore, the Gaussian curvature of M at the origin is given by

$$\begin{aligned} K(0) &= -\frac{\Delta \log \lambda}{\lambda^2} \\ &= -\frac{4|W(f_0, f_1)(0)|^{2(1-p)} (|F_1(0)|^{1-\eta_1-\delta} \dots |F_q(0)|^{1-\eta_q-\delta})^{2p}}{||f(0)||^8}. \end{aligned}$$

Comparing this with the right-hand side of (4.2), we have

$$R^{1-p} \leq C_0^{1-p} \frac{|F_1(0)|^{1-\eta_1-\delta} \dots |F_q(0)|^{1-\eta_q-\delta}}{|K(0)|^{1/2} ||f(0)||^{2(1+p)/p}}.$$

Since $|F_j|/||f|| \leq 1$ for $j = 1, 2, \cdots, q$ and

$$\frac{2(1+p)}{p} = 2\left(\frac{\gamma - q\delta}{2} + 1 \right) = \sum_{j=1}^{q} (1 - \eta_j - \delta),$$

we can conclude that

$$R^{1-p} \leq C_0^{1-p} |K(0)|^{-1/2}.$$

Combining this with (3.7), we complete the proof of Theorem II.

5. Proof of Theorem III

As in Theorem III, let $x = (x_1, x_2, x_3, x_4)\colon M \to \mathbf{R}^4$ be a nonflat complete minimal surface in $\mathbf{R}^4$, and $g = (g_1, g_2)\colon M \to P^1(\mathbf{C}) \times P^1(\mathbf{C})$ be the Gauss map. For the proof of Theorem III, we may assume that M is biholomorphic with the unit disc in $\mathbf{C}$ as in the previous sections. Take a reduced representation $g_k = (g_{k0} : g_{k1})$, and set $||g_k|| = (|g_{k0}|^2 + |g_{k1}|^2)^{1/2}$ for each $g_k\colon M \to P^1(\mathbf{C})$ $(k = 1, 2)$. Then the induced metric on M is given by

$$ds^2 = 2\left(\sum_{l=1}^{4} \left|\frac{\partial x_l}{\partial z}\right|^2\right) |dz|^2 = |h|^2 ||g_1||^2 ||g_2||^2 \, |dz|^2,$$

where $h = (\partial x_1/\partial z - \sqrt{-1}\partial x_2/\partial z)/(g_{10}g_{21})$.

Consider first the case where $g_1 \not\equiv$ const. and $g_2 \not\equiv$ const. Suppose that

$$\sum_{i=1}^{q_1} \delta_{g_1}^H(\alpha_{1i}) > 2, \qquad \sum_{j=1}^{q_2} \delta_{g_2}^H(\alpha_{2j}) > 2,$$

$$\frac{1}{\sum_{i=1}^{q_1} \delta_{g_1}^H(\alpha_{1i}) - 2} + \frac{1}{\sum_{j=1}^{q_2} \delta_{g_2}^H(\alpha_{2j}) - 2} < 1,$$

for distinct points $\alpha_{11}, \cdots, \alpha_{1q_1} \in P^1(\mathbf{C})$ and distinct points $\alpha_{21}, \cdots, \alpha_{2q_2} \in P^1(\mathbf{C})$. By Definition 1.2, there exist nonnegative constants $\eta_{k1}, \cdots, \eta_{kq_k}$ and continuous functions $u_{k1}, \cdots, u_{kq_k}$ on M for each $k = 1, 2$ such that each u_{ki} is harmonic on $M \backslash f^{-1}(\alpha_{ki})$ and satisfies the conditions

(5.1) $$\gamma_k := q_k - 2 - (\eta_{k1} + \cdots + \eta_{kq_k}) > 0 \qquad (k = 1, 2),$$

(5.2) $$\frac{1}{\gamma_1} + \frac{1}{\gamma_2} < 1,$$

(5.3) $$e^{u_{ki}} \leq ||g_k||^{\eta_{ki}} \qquad (1 \leq i \leq q_k,\ k = 1, 2),$$

(5.4) for every $\varsigma \in g_k^{-1}(\alpha_{ki})$ there exists the limit

$$\lim_{z \to \varsigma} (u_{ki}(z) - \log|z - \varsigma|) \in [-\infty, \infty).$$

Take a constant δ_0 such that $0 < q_k\delta_0 < \gamma_k$ and

$$\frac{1}{\gamma_1 - q_1\delta_0} + \frac{1}{\gamma_2 - q_2\delta_0} = 1.$$

If we choose a positive constant δ $(< \delta_0)$ sufficiently near to δ_0 and set

$$p_k := \frac{1}{\gamma_k - q_k\delta} \qquad (k = 1, 2),$$

we have

$$\text{(5.5)} \qquad 0 < p_1 + p_2 < 1, \qquad \frac{\delta p_k}{1 - p_1 - p_2} > 1 \quad (k = 1, 2).$$

Represent each α_{ki} as $\alpha_{ki} = (a^0_{ki} : a^1_{ki})$ and define holomorphic functions $F_{ki} := a^1_{ki} g_{k0} - a^0_{ki} g_{k1}$, where $|a^0_{ki}|^2 + |a^1_{ki}|^2 = 1$. Set

$$v_k := u_{k1} + \cdots + u_{kq_k},$$
$$\tilde{F}_k := F_{k1} F_{k2} \ldots F_{kq_k},$$

for each $k = 1, 2$ and define

$$v := \left(\frac{|h|\, |\tilde{F}_1|^{(1-\delta)p_1} |\tilde{F}_2|^{(1-\delta)p_2}}{(e^{v_1}|W(g_{10}, g_{11})|)^{p_1} (e^{v_2}|W(g_{20}, g_{21})|)^{p_2}} \right)^{1/(1-p_1-p_2)}.$$

The function $\log v$ is harmonic on the set

$$M' = M \backslash \{W(g_{10}, g_{11}) W(g_{20}, g_{21}) \tilde{F}_1 \tilde{F}_2 = 0\}.$$

Let $\pi \colon \tilde{M}' \to M'$ be the universal covering surface of M'. In the same manner as in §3, we can find a holomorphic function ψ on $\tilde{M}'$ such that $|\psi| = v \cdot \pi$. Define

$$w = F(\tilde{p}) = \int_{\gamma_{\tilde{p}}} \psi(z)\, dz \qquad (\tilde{p} \in \tilde{M}'),$$

as before. Then F maps an open neighborhood U of a point $\tilde{o}$ biholomorphically onto a disc Δ_R in $\mathbf{C}$, where we choose the largest R with this property. Set $\Phi := \pi \cdot (F|U)^{-1}$. Then, we have $R < \infty$ and there exists a point $a_0 \in \partial \Delta_R$ such that the image

$$\Gamma_{a_0} \colon z = \Phi(t a_0), \qquad 0 \leq t < 1,$$

of the curve $L_{a_0} = \{t a_0;\ 0 \leq t < 1\}$ by Φ tends to the boundary of M. Indeed, the same argument as in §3 is available in this case too if we use (5.5) instead of (3.4).

Now, setting $f_{kl} := g_{kl} \cdot \Phi$ and $f_k = (f_{k0} : f_{k1})$ for $k = 1, 2, \ldots$ and $l = 0, 1$, we apply the Main Lemma to the maps f_k. We then have

$$\frac{\|f_k\|^{\gamma_k - q_k\delta} e^{v_k} |W(f_{k0}, f_{k1})|}{|\tilde{F}_k|^{1-\delta}} \leq C_0 \frac{2R}{R^2 - |w|^2} \qquad (k = 1, 2),$$

where C_0 is a positive constant. On the other hand, the metric on Δ_R induced from M through Φ is given by

$$\Phi^* \, ds^2 = \left(||f_1|| \, ||f_2|| \left(\frac{|W(f_{10}, f_{11})| e^{v_1}}{|\tilde{F}_1|^{1-\delta}} \right)^{p_1} \left(\frac{|W(f_{20}, f_{21})| e^{v_2}}{|\tilde{F}_2|^{1-\delta}} \right)^{p_2} \right)^2 |dw|^2.$$

Therefore, we conclude that

$$d(0) \leq \int_{\Gamma_{a_0}} ds = \int_{L_{a_0}} \Phi^* \, ds \leq C_0^{p_1+p_2} \int_{L_{a_0}} \left(\frac{2R}{R^2 - |w|^2} \right)^{p_1+p_2} |dw| < \infty,$$

by the aid of (5.5). This contradicts the completeness of M. Thus, the proof of Theorem III(i) is complete.

We finally consider the case where $g_1 \not\equiv$ const and $g_2 \equiv$ const. Suppose that $\sum_{i=1}^{q} \delta_{g_1}^H(\alpha_i) > 3$ for distinct points $\alpha_1, \cdots, \alpha_q \in P^1(\mathbf{C})$. We can take nonnegative constants $\eta_1, \cdots, \eta_q$ with

$$\gamma := q - 2 - (\eta_1 + \cdots + \eta_q) > 1$$

and continuous functions $u_1, \cdots, u_q$ such that each u_i is harmonic on $M \backslash f^{-1}(\alpha_i)$ and satisfies conditions (C2) and (C3). Choose δ with $0 < q\delta < \gamma$ such that $p = 1/(\gamma - q\delta)$ satisfies (3.4). In this case, we use the function

$$v = \frac{|h|^{1/(1-p)} |F_1 F_2 \dots F_q|^{p(1-\delta)/(1-p)}}{e^{u_1 + \cdots + u_q} |W(g_{10}, g_{11})|^{p/(1-p)}}.$$

By the same method as before, we can construct a continuous curve of finite length which tends to the boundary of M. This contradicts the completeness of M. Thus, we complete the proof of Theorem III(ii).

References

[1] L. A. Ahlfors, *Conformal invariants*, Topics in Geometric Function Theory, McGraw-Hill, New York, 1973.

[2] C. C. Chen, *On the image of the generalized Gauss map of a complete minimal surface in* $\mathbf{R}^4$, Pacific J. Math. **102** (1982) 9–14.

[3] S. S. Chern & R. Osserman, *Complete minimal surface in euclidean n-space*, J. Analyse Math. **19** (1967) 15–34.

[4] M. J. Cowen & P. A. Griffiths, *Holomorphic curves and metrics on negative curvature*, J. Analyse Math. **29** (1976) 93–153.

[5] H. Fujimoto, *On the Gauss map of a complete minimal surface in* $\mathbf{R}^m$, J. Math. Soc. Japan **35** (1983) 279–288.

[6] ——, *Value distribution of the Gauss map of complete minimal surfaces in* $\mathbf{R}^m$, J. Math. Soc. Japan **35** (1983) 663–681.

[7] ——, *Non-integrated defect relation for meromorphic maps of complete Kähler manifolds into* $P^{N_1}(\mathbf{C}) \times \cdots \times P^{N_k}(\mathbf{C})$, Japan. J. Math. **11** (1985) 233–264.

[8] ——, *On the number of exceptional values of the Gauss map of minimal surfaces*, J. Math. Soc. Japan, **40** (1988) 235–247.

[9] D. A. Hoffman & R. Osserman, *The geometry of the generalized Gauss map*, Mem. Amer. Math. Soc. No. 236, 1980.

[10] R. Osserman, *Minimal surfaces in the large*, Comment. Math. Helv. **35** (1961) 65–76.
[11] ——, *Global properties of minimal surfaces in E^3 and E^n*, Ann. of Math. (2) **80** (1964) 340–364.
[12] ——, *A survey of minimal surfaces*, Van Nostrand Reinhold, New York, 1969.
[13] B. V. Shabat, *Distribution of values of holomorphic mappings*, Transl. Math. Monographs, Vol. 61, Amer. Math. Soc., Providence, RI, 1985.
[14] F. Xavier, *The Gauss map of a complete non-flat minimal surface cannot omit 7 points of the sphere*, Ann. of Math. (2) **113** (1981) 211–214.

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